Chapter 21

Ruelle-Takens Theorem

An immensely influential paper in the history of the study of chaos was the work of Ruelle and Takens [1], later extended with Newhouse [2] on the robustness of quasiperiodic dynamics with 3 or more incommensurate frequencies. This addressed the question of whether complexity in dynamics was likely to occur through the accumulation of Hopf-like bifurcations adding additional frequencies (and presumably spatial "modes"), or instead through the onset of low dimensional chaos. The statement of the theorem was quite mathematical, which led initially to a misunderstanding of the strength of the result in the physics community, but nevertheless it remains a powerful result.

21.1 The Theorem

> "Let v be a constant vector field on the torus $T^n = R^n/Z^n$. If $n \geq 3$ every C^2 neighborhood of v contains a vector field v' with a strange *Axiom A* attractor. If $n \geq 4$ we may take C^∞ instead of C^2."

21.1.1 Remarks

1. For the set of ODEs

$$\frac{d\vec{x}}{dt} = \vec{v}\left(\vec{x}\right) \tag{21.1}$$

   $\vec{v}(\vec{x})$ is the vector field and generates the flow $\vec{x}(t)$.

2. A constant vector field on an n-torus corresponds to a quasiperiodic flow of n incommensurate frequencies reduced to uniform rotation in each direction by a smooth transformation of the coordinates.

3. C^2 means twice differentiable; C^∞ means all derivatives can be found i.e. smooth. So the nearby functions are guaranteed to satisfy this smoothness constraint, i.e. are not unreasonably singular.

4. An Axiom A attractor is an attractor with particularly nice properties. One property is robustness under small changes of parameters.

Thus the content of the theorem is that it is possible to make arbitrarily small (but perhaps very carefully chosen) perturbations to the equations defining the motion to change the motion from n-period quasiperiodic to chaotic for $n \geq 3$. The chaotic motion remains *on* the n-torus. Furthermore the chaotic motion set up in this way cannot be destroyed by further arbitrarily small perturbations, i.e. has robustness. In a common mathematical usage, this means that a typical perturbation would lead to chaos. This led to the (mistaken) interpretation that a Hopf-like bifurcation from two frequency motion would "typically" (in a physical sense) immediately lead to chaos, since any 3-frequency torus one might anticipate being formed would suffer the fate suggested by Ruelle and Takens. The flaw in this interpretation is that the "typicality" of the required perturbations is defined in a set theoretic way (i.e. existing in every neighborhood no matter how small, which is often used as the notion of typical or generic in the math community), rather than a measure theoretic way, which is more likely to correspond to the notion of typical in a physical context.

This distinction can be understood from the analogous behavior in the phenomenon of frequency locking in the circle map. There, a quasiperiodic motion can be converted to periodic motion at a nearby rational frequency by an arbitrarily small, but carefully chosen perturbation to the parameter Ω. This periodic motion persists over a range of the parameter, and so is not destroyed by a further arbitrarily small perturbation. However for K small, the measure of quasiperiodic solutions (in Ω) is much larger than the measure of periodic solutions, so that an arbitrary small perturbation in Ω is *more likely* to lead to a (different) quasiperiodic motion than to a locked solution.

21.2 Numerical investigation

The physical likelihood of finding chaos near quasiperiodic motion on a 3-torus was investigated quantitatively in a simple example by Grebogi, Ott, and Yorke [3]. They studied a two dimensional map (corresponding to the Poincaré section of a three dimensional flow) of the form

$$\begin{aligned} \theta_{n+1} &= \theta_n + \omega_1 + \varepsilon P_1(\theta_n, \phi_n) \mod 1 \\ \phi_{n+1} &= \phi_n + \omega_2 + \varepsilon P_2(\theta_n, \phi_n) \mod 1 \end{aligned} , \tag{21.2}$$

where P_1, P_2 are nonlinear functions periodic in both θ and ϕ with period 2π. The explicit forms used were sums of sinusoidal functions $A_{rs}\sin[2\pi(r\theta + s\phi + B_{rs})]$ with A_{rs} and B_{rs} chosen randomly for (r, s) taking the values (0, 1), (1, 0), (1, 1), $(1, -1)$. Equation (21.2) takes the form of two, nonlinearly coupled circle maps.

The range of ω_1 and ω_2 leading to different types of motion was then investigated numerically for increasing values of ε corresponding to increasing nonlinearity. As well as three frequency quasiperiodic (QP) motion, there is the possibility of frequency locking to two frequency quasiperiodic motion or to periodic motion (P), as well as chaos. The types of motion were identified through the values of the two Lyapunov exponents (together with the third value 0 for the corresponding flow):

Map exponents	Flow Exponents	Dynamics
0, 0	0, 0, 0	3-frequency QP
0, −	0, 0, −	2-frequency QP
−, −	0, −, −	1-frequency (P)
+, ?	+, 0, ?	Chaotic

.

They found the percentage of the values of leading to each type of motion

Attractor	$\varepsilon/\varepsilon_c = 0.375$	$\varepsilon/\varepsilon_c = 0.75$	$\varepsilon/\varepsilon_c = 1.125$
3-frequency QP	82%	44%	0%
2-frequency QP	16%	38%	32%
Periodic	2%	11%	31%
Chaotic	0%	7%	36%

where ε_c is the value of ε for which the map becomes noninvertible. (When the map is noninvertible it can be shown that there is no 3-frequency periodic motion, c.f. the 1d circle map at $K > 1$.) Thus in this example at least, 3-frequency quasiperiodic motion typically survives quite strong nonlinearity, and the fate when this dynamics breaks down, even for large nonlinearity, is as likely to be locked (quasiperiodic or periodic) motion as chaotic motion. You can investigate these results in the demonstration.

21.3 Experiment

Experimentally 3-frequency quasiperiodic motion has been documented by Gollub and Benson [4], and 4-frequency and 5-frequency quasiperiodic motion by Walden et al.[5], both in fluid convection.

February 27, 2000

Bibliography

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