Chapter 2

The Language of Dynamical Systems

The well known example of the driven, damped pendulum provides a convenient introduction to some of the language of dynamical systems.

2.1 The ideal pendulum

If we define θ as the angular displacement of the pendulum from the equilibrium (hanging down) position, the equation of motion for the oscillations of an ideal pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0, \tag{2.1}$$

where l is the length and g is the gravitational acceleration. We can write (2.1) as two first order equations

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\frac{g}{l}\sin\theta \end{aligned} \tag{2.2}$$

introducing the angular velocity ω, and then can use (θ, ω) as our phase space coordinates. Later, we will introduce a different pair of coordinates, using the angular momentum $J = Ml^2\omega$ as the second coordinate (with M the mass of the pendulum). The dynamics in the phase space is given by a series of trajectories, as shown in the figure: Since there is no dissipation in the equations, the energy is conserved, and we can imagine labelling each trajectory by its energy.

Various features are marked on the figure

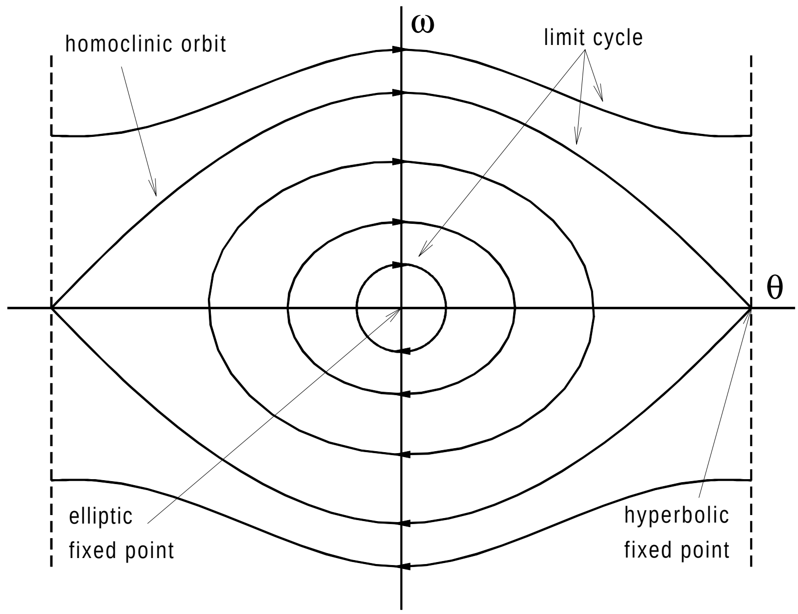

Figure 2.1: Phase space of the ideal pendulum

The rest position $\theta = \omega = 0$ is called a “fixed point”. This is an “elliptic” fixed point, since nearby orbits take the form of ellipses (or circles in scaled coordinates). Naively we might call this a stable fixed point, but since there is no dissipation perturbations from the fixed point do not decay back to the fixed point.

The θ coordinate runs from $-\pi$ to π. There is a second fixed point at $(\pi, 0)$ corresponding to the pendulum pointing vertically up: this is a “hyperbolic” fixed point, because nearby trajectories take this form. These trajectories take an initial point near the fixed point far away, and we would naively call this an unstable fixed point.

The remaining orbits are periodic in time, and are called “limit cycles”. For small energies, near $(0, 0)$ the limit cycles are the familiar simple harmonic motion, represented by circles or ellipses (stretched circles) in the phase space. These would give a single peak in a power spectrum, and would sound like a pure musical tone. For larger energies, the orbit becomes distorted in the phase space and are no longer simple harmonic. The power spectrum would show harmonics, with additional frequencies at multiples of the fundamental, and the tone, although representing one musical note, would sound more complex.

A special pair of orbits leave the hyperbolic fixed point, and then eventually return to it. (Remember the θ coordinate wraps around!) These are known as "homoclinic" orbits. The dynamics slows down approaching the fixed point, and the period of the limit cycle orbits diverge as their energy approaches the energy of the homoclinic orbit. (In other systems we might have a "heteroclinic" orbit connecting two different hyperbolic fixed points.)

We know that the ideal pendulum is a Hamiltonian system. This means we can use the energy to construct a Hamiltonian :

$$H = \frac{1}{2I} J^2 + Mgl(1 - \cos\theta) \tag{2.3}$$

which is just the energy written as a function of the two "canonically conjugate" variables, the angular position θ and the angular momentum $J = I\omega$ with $I = Ml^2$ the moment of inertia. The Hamiltonian formulation of the dynamics is then

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial J} = \frac{J}{I} \\ \dot{J} &= -\frac{\partial H}{\partial \theta} = -Mgl\sin\theta \quad . \end{aligned} \tag{2.4}$$

It is easy to see that these are the same as (2.2).

A very important property of Hamiltonian systems is that the dynamics "preserves volumes in phase space". This means that if we start off many copies of the system, with initial conditions filling some small volume in phase space, then as the system evolves the volume of phase space containing the evolving points distorts in shape, but keeps a fixed volume.

We first define a velocity in phase space giving the time dependence of the phase space coordinates, here

$$\vec{V} = (\dot{\theta}, \dot{J}). \tag{2.5}$$

Now it is easy to verify from the equations of motion that this "velocity" is divergence free:

$$div\vec{V} \equiv \frac{\partial\dot{\theta}}{\partial\theta} + \frac{\partial\dot{J}}{\partial J} = 0. \tag{2.6}$$

This in fact is a general consequence of the form of the Hamilton equations of motion. Just as for an incompressible fluid, this is equivalent to volume conserving flow, as can be seen by integrating over an arbitrary volume and using Gauss's theorem.

An immediate consequence of this result is that there are no attractors in Hamiltonian systems: there can be no attracting fixed point to which initial conditions distributed over some volume converge, since this would yield a volume of points in phase space contracting asymptotically to zero.

You can investigate the phase space of the ideal pendulum in demonstration 1.

2.2 The dissipative pendulum

If we add a dissipative force proportional to the velocity, the equation of motion becomes

$$\frac{d^2\theta}{dt^2} + \eta \frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0 \quad . \tag{2.7}$$

It is easy to see that almost all phase space trajectories spiral into the fixed point at $(0, 0)$. This is now truly a "linearly stable" fixed point, since if a small perturbation is made from the fixed point, the perturbation decays in time (in fact exponentially for small enough perturbations). On the other hand the fixed point at $(\pi, 0)$ is "linearly unstable" because a small perturbation from this fixed point grows exponentially. Only very carefully tuned initial conditions will lead to a trajectory ending on the unstable fixed point, and almost all perturbations to the initial condition will lead to a trajectory that may approach close to the unstable fixed point, but eventually spirals into the stable fixed point. The $(0, 0)$ fixed point is "attracting", and in this case the "basin of attraction" i.e. the set of initial conditions leading to trajectories that approach the fixed point, is the whole phase space except for points on the "stable manifold" of the hyperbolic fixed point, which is a set of zero area in the phase space.

The dynamical behavior can be studied in demonstration 2.

2.3 The periodically driven, damped pendulum

The situation is more interesting if we also drive the pendulum, feeding in energy to resupply the energy dissipated. Simple harmonic driving leads to the following equation

$$\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \sin\theta = g\cos(\omega_D t) \tag{2.8}$$

where we have rescaled time so that the period of small oscillations of the undamped and undriven pendulum is unity, and we have written the scaled dissipation coefficient as γ.

For small amplitudes of driving g, and assuming a small initial condition, we can replace $\sin\theta$ by θ and solve the equation analytically:

$$\theta = \frac{g}{\sqrt{\left(1-\omega_D^2\right)^2+\gamma^2\omega_D^2}}\cos(\omega_D t+\phi)+Ae^{-\gamma t/2}\cos(\omega t+\phi_A) \tag{2.9}$$

with

$$\tan\phi = -\frac{\gamma\omega_D}{\left(1-\omega_D^2\right)}, \quad \omega=\sqrt{1-\frac{\gamma^2}{4}}. \tag{2.10}$$

This is the well known resonant response (the first term) oscillating at the applied frequency, together with decaying free oscillations (the second term) depending on the initial conditions. We would call this solution an attracting limit cycle.

What happens for large driving amplitudes? Here there are no analytic solutions, and we must proceed numerically. To gain some intuition we would like to view the dynamics in a phase space. To this we convert the equation to autonomous form by using *three* variables

$$\begin{aligned} \dot\theta &= \omega \\ \dot\omega &= -\gamma\omega - \sin\theta + g\cos(\theta_D) \\ \dot\theta_D &= \omega_D \end{aligned} \tag{2.11}$$

where we have introduced the "phase of the driving" θ_D. This method of gaining an autonomous form at the expense of an extra equation is a common and useful trick. We again have a three dimensional phase space as in the Lorenz model: do we find chaos?

First it is useful to look again at volumes in phase space. Now we have for the divergence of the velocity $\vec V = (\dot\theta, \dot\omega, \dot\theta_D)$

$$div\vec V = \frac{\partial\dot\theta}{\partial\theta}+\frac{\partial\dot\omega}{\partial\omega}+\frac{\partial\dot\theta_D}{\partial\theta_D} = -\gamma \qquad , \tag{2.12}$$

a constant! This means that volumes contract at a constant proportional rate γ. (The Lorenz model shows this special feature too: there the proportional contraction rate is $\sigma+1+b$). Systems whose phase space volumes are not conserved, and on some sort of average contract, are called dissipative systems. At first sight we might expect a volume of initial conditions must contract to a point, i.e. all orbits approach stable fixed points asymptotically—not very interesting. However this is not the only possibility. We already know from the small amplitude case that the orbits may approach an attracting limit cycle. Even more interesting, a phase space volume may be stretched in one or more directions, whilst it is contracting in the remaining ones so that overall the volume contracts. This is the crude description of how chaos may occur in purely contracting dissipative systems. How chaos occurs in perhaps this simplest and most familiar dynamical system is illustrated in demonstrations 3-7.

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