Chapter 4

One dimensional Maps

The ordinary differential equation studied in chapters 1-3 provide a close link to actual physical systems—it is easy to believe these equations provide at least an approximate description of phenomenon in the real world. However it turns out to be hard to prove mathematical results for such systems, and numerical evolution with sufficient accuracy and over sufficient times to accumulate large amounts of data for statistical analysis remains a limitation. As a result the study of "maps" has played an important role in the understanding of chaos. In this chapter the study of "one dimensional maps" is introduced, and in the next chapter "two dimensional maps" are described.

4.1 Flows and Maps

4.1.1 Flows

The framework we have studied so far can be formalized as an set of ordinary differential equations

$$\dot{U} = f(U; r) \tag{4.1}$$

where U is a vector of phase space coordinates of dimension N, and r is a vector of control parameters. The equations are:

- autonomous: no time appears on the right hand side;
- deterministic: evolution is completely specified (by instantaneous U), there is no stochasticity in the equations;

- dissipative: (for non-Hamiltonian systems) N dimensional volumes in phase space shrink to a lower dimensional phase space giving attractors in phase space.

The mathematical structure is that of smooth *vector fields* in R^N, and the solution $U(t)$, $U(t_0) = U_0$, is called a flow. This mathematical structure yields useful theorems [1], for example the *Poincaré-Bendixson theorem*, which tells us that the only long time asymptotic flows in two dimensions ($N = 2$) are fixed points, limit cycles, and homoclinic or heteroclinic orbits. One consequence is that there is no chaos in a two-dimensional phase space.

4.1.2 Maps

Maps give an evolution analogous to (4.1) but with a discrete "time"

$$U_{n+1} = F(U_n; r) \tag{4.2}$$

where U_n is an N-dimensional vector with components $U_n^{(i)}$ and F is a map from R^N onto R^N. These evolution equations are again autonomous and deterministic. The effect of the evolution on volumes in phase space is given by the Jacobean J

$$J = \left| \det \frac{\partial F^{(i)}}{\partial U^{(j)}} \right| \quad . \tag{4.3}$$

If $J = 1$ volumes are preserved under the iteration and the map is called *volume* or *area preserving*; if $J < 1$ (on average) volumes decrease and the map is *dissipative*.

There are a number of ways to connect maps with flows:

- We can integrate the flow for times $n\tau$ with τ some chosen fixed interval.
- The map can be the $N-1$ dimensional Poincaré section of an N dimensional flow.

  In these two cases a smooth flow gives a smooth map with a smooth, unique inverse (because we can integrate the flow backwards in time over the finite interval τ) i.e. is a *diffeomorphism*.

- We can construct a "one dimensional return map" as illustrated in chapter 1. In this case the map is an idealization, and need not be invertible.

Alternatively, the modeling of the physical system may be most appropriate in terms of a discrete time, for example population dynamics may be best described in terms of annual populations.

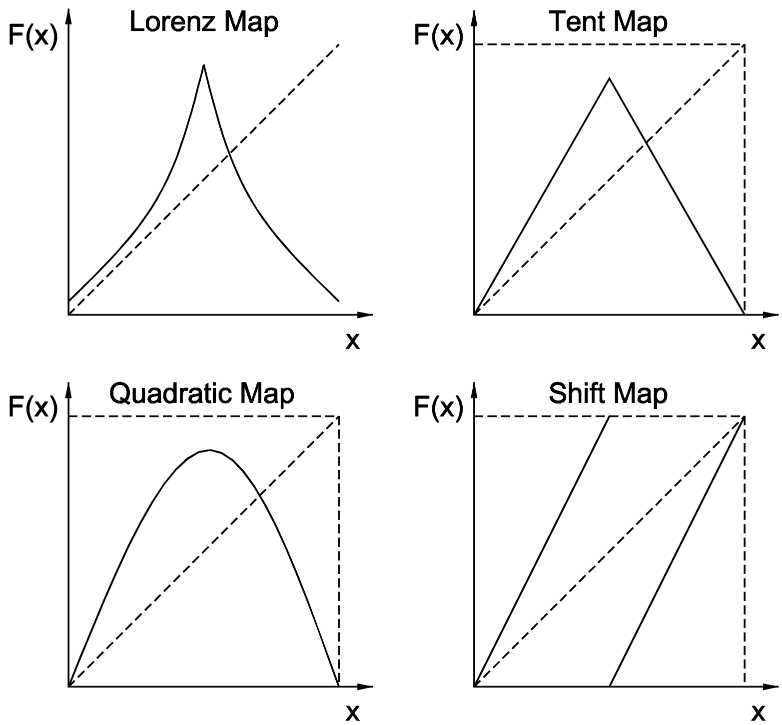

Figure 4.1: Some One Dimensional Maps

4.2 Examples of 1-d maps

Some examples of one dimensional maps are listed here and shown in figure 4.1. Many are motivated in terms of their simplicity.

1. The Lorenz map: given by plotting $Z_{n+1}^{\max}$ against $Z_n^{\max}$ where $Z_n^{\max}$ is the value of the Z coordinate at its nth maximum. The empirical form of the map) for Lorenz's parameter choice is sketched in figure 1: there is no analytic expression.

2. The tent map:

$$F(x) = \begin{cases} ax & x < \frac{1}{2} \\ a(1-x) & x > \frac{1}{2} \end{cases} \tag{4.4}$$

for $0 \leq a \leq 2$.

3. The quadratic or logistic map

$$F(x) = ax(1-x) \tag{4.5}$$

for $0 \le a \le 4$.

4. The shift map:

$$F(x) = 2x \mod 1 \tag{4.6}$$

5. The sine map

$$F(x) = \frac{a}{4}\sin(\pi x) \tag{4.7}$$

for $0 \le a \le 4$.

It is conventional to scale variables so that the interval $0 \le x \le 1$ is mapped into itself. The ranges of the control parameter a quoted are the limits for when this condition is satisfied.

4.3 Iterating 1-d maps

The iteration of one dimensional maps is easy to see graphically: if we plot $y = F(x)$ and $y = x$ the iterations are given by successive steps between these two curves:

$$y = F(x_n) \qquad x_{n+1} = y \tag{4.8}$$

Successive iterations from a given initial values are given by successive operations of the map F, an operation known as "functional composition":

$$\begin{array}{rclcl} x_1 & = & F(x_0) & & \\ x_2 & = & F(x_1) & = & F(F(x_0)) \\ & \vdots & & & \\ x_n & = & F(x_{n-1}) & = & F(F \ldots F(x_0)) \end{array} . \tag{4.9}$$

The (somewhat confusing) notation $F^2(x)$ is used for $F(F(x))$ i.e. the order 2 functional composition. Note this is not the square $(F(x))^2$! We can study every 4th (for example) iteration of F by iterating F^4, etc.

It should be evident from the graphical scheme that the intersection x_f of $y = F(x)$ with $y = x$ is a fixed point of the iteration i.e.

$$F(x_f) = x_f. \tag{4.10}$$

We can easily answer the question of whether an initial condition close to x_f approaches the fixed point under iteration (when we call the fixed point stable) or moves away from it (unstable fixed point) by linearizing the evolution about x_f: write $x = x_f + \delta x$ and then using a Taylor expansion with $F'(x)$ the derivative of the function

$$x_{n+1} = x_f + \delta x_{n+1} = F(x_x + \delta x_n) \simeq F(x_f) + \delta x_n F'(x_f) \tag{4.11}$$

so that

$$\delta x_{n+1} = F'(x_f)\delta x_n \tag{4.12}$$

and $|\delta x_n|$ will increase on successive iterations for $\left|F'(x_f)\right| > 1$. Thus the fixed point is stable for $\left|F'(x_f)\right| < 1$ and is unstable for $\left|F'(x_f)\right| > 1$.

In the quadratic map, when the fixed point x_f is stable almost all initial conditions lead to an orbit that converges to the fixed point ($x = 0$ and $x = 1$ being exceptional initial conditions). What happens when x_f becomes unstable (which happens at $a = 3$)? For this map, for nearby values of a the orbits converge to an orbit which alternately visits two values x_1 and x_2: this is the discrete time version of a limit cycle or periodic orbit (here period 2). The second iterate function $y = F^2$ yields three intersections with the line $y = x$. It is easy to check that at two of these the magnitude of the slope $\left|dF^2/dx\right|$ is less than unity, i.e. there are two stable fixed points of F^2, and these correspond to x_1 and x_2. The third fixed point of F^2 is unstable: x_f is of course an unstable fixed point of F^2.

These points are illustrated in demonstrations 1-2.

4.4 Bifurcations in 1-d maps

As the map parameter a is changed, the character of the long time solution may dramatically change, from a fixed point to a period two limit cycle for example. These changes are called *bifurcations*. The bifurcations that occur, and the different types of orbits, are well shown by the “bifurcation map”. This is constructed with the parameter a along the abscissa, and all values of x visited (after some number of iterations to eliminate transients) plotted as points along the ordinate. A fixed point orbit over a range of a appears as a single curve, which splits into two curves at the bifurcation to the period 2 orbit etc. Chaotic dynamics, where the orbit visits an infinite number of points (otherwise the orbit would repeat, and therefore be periodic) appears as bands of continua of points (subject to limitations of how many points are actually plotted in the implementation).

The bifurcation maps of our one dimensional maps show that even these very simple dynamical systems show an amazingly rich bifurcation structure (demonstration 3). This complexity in the quadratic map was studied by May [2] in the context of population dynamics.

4.5 Invariant measure

Since we cannot expect to know the chaotic dynamics precisely, we need a statistical description. From the bifurcation map it appears that for some parameter values the iterated points cover intervals of the line with some density or probability distribution. We can use this to define the "invariant measure" of the attractor. We first define the measure $\rho(x, x_0)$ as the density of points at x given by iterating many times from the initial point x_0, i.e.

$$\rho(x, x_0)dx = \lim_{N\to\infty} \frac{1}{N} \times\ n(x, x_0), \tag{4.13}$$

with $n(x, x_0)$ the number of $\left\{F^N(x_0), F^{N-1}(x_0)\ldots F(x_0)\right\}$ in length dx about x, or perhaps

$$\rho(x, x_0) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta(x - F^i(x_0)), \tag{4.14}$$

although in this last compact form we may need to fuzz out the delta function until after the $N \to \infty$ limit is taken, since, for example, if the attractor is a fixed point (where the measure should be concentrated at the fixed point) the orbit never exactly reaches the fixed point for any finite N.

We would hope that the measure is a property of the attractor, and is independent of the choice of initial point x_0, except for a number of "bad choices" such as at unstable fixed points or unstable periodic orbits. Thus we would hope that $\rho(x, x_0)$ is independent of x_0 for "almost all" x_0, i.e. except for a set of measure zero (with respect to the physical Lebesgue measure of the unit interval), and then defines the measure of the attractor $\rho(x)$. By construction this measure is "invariant" i.e. is unchanged by iterating the map variable: if $y = F(x)$ then $\rho(y)dy = \rho(x)dx$ since all points in the interval dx end up in the interval $dy = F'(x)dx$. The definition also incorporates the notion of ergodicity, i.e. "time averages equal measure averages" for almost all initial conditions (with respect to the initial Lebesgue measure).

An alternative path, which does not make reference to a second measure for discussing the initial conditions, is to focus on the "invariant measures", i.e. measures that are left unchanged by the dynamics. There are typically an infinite number of invariant measures (again, a measure concentrated on an unstable fixed point or orbit is invariant): some process is needed to select the physical measure that will presumably be useful experimentally or numerically. A physically appealing process, attributed to Kolmogorov, is to add a small amount of random noise to the dynamics, which will usually yield a unique measure, and then to let the noise strength tend to zero. Other schemes, which have advantages in constructing mathematical proofs, have also been discussed, for example see [1] which provides a useful discussion of the interplay of mathematical and physical ideas. An invariant measure that cannot be decomposed into different pieces that are themselves invariant is said to be ergodic. The ergodic theorem then tells us that time averages can be replaced by measure averages, again for almost all initial conditions, but now almost all means except for a set of measure zero with respect to the invariant measure.

In certain cases the invariant measure can be constructed directly from its definition. Consider a density of points after n iterations $\rho_n(x)$. Then under iteration of the map the density evolves according to the Frobenius-Perron equation:

$$\rho_{n+1}(y) = \int dx\, \delta\left[y - F(x)\right] \rho_n(x) \tag{4.15}$$

which implements the idea that all points in the interval dx end up in the interval dy where $y = F(x)$ and $dy = F'(x)dx$. The invariant measure is given by equating ρ_n and ρ_{n+1}, and the approach to the invariant measure from an initial density $\rho_0(x)$ can also be studied.

The invariant measure often shows considerable structure. For example the quadratic map at $a = 4$ has square root singularities at the endpoints: $\rho(x) \propto 1/\sqrt{x(1-x)}$, and at values of a intermediate between 0 and 1 shows a rich structure of singularities (see demonstration 4).

4.6 Lyapunov exponents

The idea of the instability of a fixed point can be generalized to make the idea of "sensitive dependence on initial conditions" more quantitative. Equation 4.12 can in fact be used for the expansion of a small separation at any x_n

$$\delta x_{n+1} = F'(x_n)\delta x_n \tag{4.16}$$

so that the product of the derivatives at successive iterations gives us the expansion (or contraction) of the separation between the iterates of nearby points.

More precisely we start with initial conditions x_0 and $x_0 + \varepsilon$ and ask for the distance between the nth iterates, which we expect to grow as

$$|\delta x_n| = \varepsilon e^{n\lambda(x_0)} \tag{4.17}$$

where $\lambda(x_0)$ is the Lyapunov exponent for the initial condition x_0, i.e.

$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\varepsilon\to 0} \frac{1}{n} \log \left| \frac{F^n(x_0+\varepsilon) - F^n(x_0)}{\varepsilon} \right| = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{dF^n(x_0)}{d(x_0)} \right| . \tag{4.18}$$

For systems with an ergodic invariant measure the limit exists and is independent of the initial condition x_0 for almost all initial conditions (e.g. not those points exactly on unstable periodic orbits), and will be denoted λ and called the *Lyapunov exponent* of the map. The derivative can be evaluated by the chain rule in terms of derivatives of F at the intermediate iterations

$$\frac{dF^n(x_0)}{d(x_0)} = F'(x_{n-1})F'(x_{n-2})\ldots F'(x_1)F'(x_0). \tag{4.19}$$

Thus we can compactly write

$$\lambda = \left\langle \log \left| F' \right| \right\rangle \tag{4.20}$$

where the average $\langle\rangle$ is over the iterations of the map.

A positive value of λ corresponds to the difference between closely spaced initial conditions growing (on average exponentially) with iteration i.e. to sensitive dependence on initial conditions. Thus a positive Lyapunov exponent is a signature of chaos, and may be used as a defining criterion.

The Lyapunov exponent of the tent map is easily calculated since $\left|F'\right| = |a|$ for all values of x. Thus $\lambda = \log|a|$ and we expect chaotic dynamics for $1 < a \leq 2$. The Lyapunov exponent $\lambda(a)$ for other maps is shown in demonstrations 5-6.

Bibliography

[1] J-P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985)

[2] R.M. May, Nature **261**, 459 (1976)