Stability of Synchronized Chaos in Coupled Dynamical Systems

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Abstract

We consider the stability of synchronized chaos in coupled map lattices and in coupled ordinary differential equations. Applying the theory of Hermitian and positive semidefinite matrices we prove two results that give simple bounds on coupling strengths which ensure the stability of synchronized chaos. Previous results in this area involving particular coupling schemes (e.g. global coupling and nearest neighbor diffusive coupling) are included as special cases of the present work.

PACS Numbers: 05.40.+j, 05.45.+b

I. INTRODUCTION

Synchronization of coupled chaotic systems [1,2] has found applications in a variety of fields including communications [1,3], optics [4,5], neural networks [6–8] and geophysics [9]. An essential prerequisite for these applications is to know the bounds on the coupling strengths so that the stability of the synchronous state is ensured. Previous attempts [10–29] aimed at obtaining such conditions have typically looked either at systems of very small size or at very specific coupling schemes (diffusive coupling, global all to all coupling etc. with a single coupling strength). More recently, Pecora and Carroll [30] introduced the notion of a master stability function for general coupling topologies, but this function can only be accessed in a numerical fashion. The contribution of this Letter is a methodology that can lead to analytical bounds on the individual coupling strengths under the mild assumption that the coupling be symmetric. We demonstrate the method in the form of two theorems, one applicable to coupled map lattices and the other to coupled ordinary differential equations (ODEs).

II. STABILITY RESULTS

The coupled map lattices we consider are in the form

$$\mathbf{x}^i(n+1) = \mathbf{f}(\mathbf{x}^i(n)) + \frac{1}{L}\sum_{j=1}^{L} a_{ij}\left[\mathbf{f}(\mathbf{x}^j(n)) - \mathbf{f}(\mathbf{x}^i(n))\right], \tag{1}$$

and we use

$$\dot{\mathbf{x}}^i(t) = \mathbf{f}(\mathbf{x}^i(t)) + \sum_{j=1}^{L} b_{ij}\mathbf{x}^j(t), \tag{2}$$

for coupled ODEs. Here $\mathbf{x}^i$ is an M-dimensional state vector describing the ith map/ODE and $\mathbf{f}$ is an M-dimensional map/vector field. We assume that the coupling is symmetric, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, and that, in the absence of coupling, the individual M-dimensional system is chaotic with the largest Lyapunov exponent $h_{max} > 0$. With the additional constraint that $\sum_{j=1}^{L} b_{ij} = 0$ it is easy to see that the synchronized state, $\mathbf{x}^1 = \mathbf{x}^2 = \cdots = \mathbf{x}^L = \mathbf{x}$, is a solution to our models. We obtain stability conditions by requiring that the "transverse" Lyapunov exponents (which will be defined later) are all negative. We note that the stability obtained here is in the sense of weak stability (or Milnor stability), which ensures the transverse stability of typical orbits, and is a necessary condition for asymptotic stability where all orbits are transversely stable. Our stability results for the synchronized chaotic solution are expressed in the following two theorems.

Theorem 1. For Eq. (1) the synchronized chaotic state is stable if for all $i, j,\ j \neq i$,

$$[1 - \exp(-h_{max})] < a_{ij} < [1 + \exp(-h_{max})]. \tag{3}$$

Theorem 2. For Eq. (2) the state of synchronized chaos is stable if

$$b_{ij} > h_{max}/L, \quad \forall\ i, j,\ j \neq i. \tag{4}$$

We now sketch the proof of Theorem 1. Linearizing Eq. (1) around the synchronized chaotic state $\mathbf{x}$ we get :

$$\mathbf{z}^i(n+1) = \mathbf{J}(\mathbf{x}(n)) \left[\mathbf{z}^i(n)(1 - \frac{1}{L}\sum_{j\neq i} a_{ij}) + \frac{1}{L}\sum_{j\neq i} a_{ij}\mathbf{z}^j(n) \right], \tag{5}$$

where $\mathbf{z}^i$ denotes the ith map's deviations from $\mathbf{x}$ and $\mathbf{J}$ is the $M \times M$ Jacobian matrix. In terms of the $M \times L$ state matrix $\mathbf{S}(n) = (\mathbf{z}^1(n)\ \mathbf{z}^2(n)\ \ldots\ \mathbf{z}^L(n))$ Eq. (5) can be written as the following matrix equation:

$$\mathbf{S}(n+1) = \mathbf{J}(\mathbf{x}(n))\mathbf{S}(n)\mathbf{C}^T, \tag{6}$$

where $\mathbf{C}^T$ is the transpose of the $L \times L$ coupling matrix $\mathbf{C}$ containing the coupling coefficients:

$$\begin{aligned} [\mathbf{C}]_{ii} &= 1 - \frac{1}{L}\sum_{j\neq i} a_{ij}, \quad i = 1, 2, \ldots, L, \\ [\mathbf{C}]_{ij} &= a_{ij}/L, \quad i \neq j \end{aligned} \tag{7}$$

Since the coupling coefficients are assumed to be real and symmetric, $\mathbf{C}$ is Hermitian. It can be diagonalized and all its eigenvalues are real [31]: $\mathbf{C} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1}$. Here $\mathbf{E}$ and $\mathbf{\Lambda}$ are the eigenvector and eigenvalue matrices of $\mathbf{C}$, respectively. Let $\mathbf{e}$ be one of the eigenvectors of $\mathbf{E}$ and λ its associated eigenvalue. Acting Eq. (6) on $\mathbf{e}$ we get:

$$\mathbf{S}(n+1)\mathbf{e} = \lambda\mathbf{J}(\mathbf{x}(n))\mathbf{S}(n)\mathbf{e}. \tag{8}$$

Let $\mathbf{u}(n) = \mathbf{S}(n)\mathbf{e}$. Then

$$\mathbf{u}(n+1) = \lambda\mathbf{J}(\mathbf{x}(n))\mathbf{u}(n). \tag{9}$$

We now compute the Lyapunov exponents for the above reduced system. We note that $\lambda = 1$ is always an eigenvalue of $\mathbf{C}$ and its corresponding eigenvector is $(1\ 1\ \ldots 1)^T$. In this case, the above equation is just the linearization of the individual map which was assumed to be chaotic. Therefore, the eigenvector $(1\ 1\ \ldots 1)^T$ of $\mathbf{C}$ with eigenvalue 1 corresponds to the synchronized chaotic state. The Lyapunov exponents in this case are nothing but the Lyapunov exponents for the individual system. Hence they are given by $h_1 = h_{max}$, h_2, ... , h_M. These describe the dynamics within the synchronization manifold defined by $\mathbf{x}^i = \mathbf{x}\ \forall i$.

Next we consider the remaining eigenvalues and eigenvectors. Since $\mathbf{C}$ is a symmetric matrix, the remaining eigenvectors span a $(L-1)$-dimensional subspace orthogonal to the eigenvector $(1\ 1\ \ldots 1)^T$. Consequently, this subspace is orthogonal to the synchronization manifold. For each $\lambda \neq 1$ we calculate the Lyapunov exponents for Eq. (9). Since λ is a real number ($\mathbf{C}$ being a symmetric matrix), the Lyapunov exponents are easily calculated. Denoting them by $\mu_1(\lambda)$, $\mu_2(\lambda)$, ..., $\mu_M(\lambda)$, we have

$$\mu_i(\lambda) = h_i + \ln|\lambda|, \quad i = 1, 2, \ldots, M. \tag{10}$$

We refer to these Lyapunov exponents as transversal Lyapunov exponents [11] since they characterize the behavior of infinitesimal vectors transversal to the synchronization manifold. These determine the stability of the synchronized chaotic state. For stability, we require the transversal Lyapunov exponents for each $\lambda \neq 1$ to be negative. This is equivalent to the statement

$$\mu_{max}(\lambda) = h_{max} + \ln|\lambda| < 0. \tag{11}$$

In other words, we require $|\lambda| < \exp(-h_{max})$ for each $\lambda \neq 1$. From this equation we see that for the synchronized chaotic state to be stable, $\lambda = 1$ should be the eigenvalue of $\mathbf{C}$ with the largest magnitude. Ordering the eigenvalues of $\mathbf{C}$ as $\lambda_1 = 1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_L$, the stability conditions can be rewritten as

$$\lambda_2 < \exp(-h_{max}), \tag{12}$$
$$\lambda_L > -\exp(-h_{max}). \tag{13}$$

Our goal is to obtain bounds on a_{ij} such that the above two inequalities are simultaneously satisfied. Consider a Hermitian matrix $\mathbf{K}$ defined as follows:

$$\begin{aligned}[][\mathbf{K}]_{ii} &= 1 - \frac{(L-1)R}{L}, \quad i = 1, 2, \ldots, L,\\ [\mathbf{K}]_{ij} &= \frac{R}{L}, \quad i \neq j,\end{aligned} \tag{14}$$

where R is a constant which will be characterized later. Consider the matrix $\mathbf{P} = \mathbf{K} - \mathbf{C}$. We see that the diagonal elements are given by

$$\mathbf{P}_{ii} = \frac{1}{L}\sum_{j\neq i} a_{ij} - \frac{(L-1)R}{L}, \quad i = 1, 2, \ldots, L. \tag{15}$$

These are positive if $a_{ij} > R\,\forall\, i, j,\ j \neq i$. Next consider the absolute value of the off-diagonal elements of $\mathbf{P}$:

$$|\mathbf{P}_{ij}| = |\frac{R}{L} - \frac{a_{ij}}{L}|, \quad \forall\, i, j,\ j \neq i. \tag{16}$$

If $a_{ij} > R$, it can be seen that

$$|\mathbf{P}_{ii}| \geq \sum_{j\neq i} |\mathbf{P}_{ij}|, \quad i = 1, 2, \ldots, L. \tag{17}$$

This implies that $\mathbf{P}$ is positive semidefinite [31].

We now introduce the concept of positive semidefinite ordering [31]. Since Hermitian matrices are generalizations of real numbers and positive semidefinite matrices are generalizations of nonnegative real numbers, one can introduce an ordering among Hermitian matrices as follows [31]: Let $\mathbf{A}$, $\mathbf{B}$ be $L \times L$ Hermitian matrices. We write $\mathbf{A} \succeq \mathbf{B}$ if the matrix $\mathbf{A} - \mathbf{B}$ is positive semidefinite. Further, if $\mathbf{A} \succeq \mathbf{B}$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_L$ are the ordered eigenvalues of $\mathbf{A}$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_L$ are the ordered eigenvalues of $\mathbf{B}$, then

$$\alpha_i \geq \beta_i, \quad i = 1, 2, \ldots, L. \tag{18}$$

Earlier, we had already shown that $\mathbf{P} = \mathbf{K} - \mathbf{C}$ is positive semidefinite if $a_{ij} > R\,\forall\, i, j\ j \neq i$. Since $\mathbf{K}$ and $\mathbf{C}$ are Hermitian, we have $\mathbf{K} \succeq \mathbf{C}$. Now the largest eigenvalues of both $\mathbf{K}$

and $\mathbf{C}$ are equal to 1. The second largest eigenvalue of $\mathbf{K}$ can be easily calculated and is equal to $1 - R$. Therefore $\mathbf{K} \succeq \mathbf{C}$ implies that

$$\lambda_2 \leq (1 - R). \tag{19}$$

Comparing with the inequality given in Eq. (12), we see that this constraint is obeyed if $(1 - R) < \exp(-h_{max})$. That is,

$$R > 1 - \exp(-h_{max}). \tag{20}$$

But $\lambda_2 \leq 1 - R$ only if $a_{ij} > R \; \forall \; i, j \; j \neq i$. Putting the two inequalities together, we get the first stability condition:

$$a_{ij} > [1 - \exp(-h_{max})], \quad \forall \; i, j, \; j \neq i. \tag{21}$$

Next we need to satisfy the second stability constraint given in Eq. (13). Following a procedure similar to the one above, we get the second stability condition:

$$a_{ij} < [1 + \exp(-h_{max})], \quad \forall \; i, j, \; j \neq i. \tag{22}$$

Combining the stability conditions given in Eqs. (21) and (22) we get our final result as follows. The synchronized chaotic state of Eq. (1) is stable (in the Milnor sense) if the coupling coefficients a_{ij} (which are assumed to be symmetric) obey the stability condition given in Eq. (3). Note that as $h_{max} \to 0$, the range of allowed coupling strengths increases reaching a maximum range of $(0, 2)$. As $h_{max} \to \infty$, the range decreases to zero. This confirms the intuitive expectation that it should be harder to stabilize a synchronized state which is more chaotic. The above result generalizes earlier results [11,13] obtained assuming that all coupling coefficients are identical. Further, in some of the earlier papers [11], it was assumed that the constant (dimensionless) coupling strength is less than 1 and hence the upper bound given here (which is greater than 1) was not explicitly observed in those previous studies.

We now numerically verify the above result by studying a system of 100 Henon maps. The f [cf. Eq. (1)] for an individual Henon map is given by:

$$f_1(x_1, x_2) = 1 + x_2 - a x_1^2; \quad f_2(x_1, x_2) = b x_2. \tag{23}$$

For the values $a = 1.4$ and $b = 0.3$, the maximum Lyapunov exponent h_{max} is found to be 0.43. We now couple 100 Henon maps using the scheme given in Eq. (1) where we randomly generate the coupling strengths a_{ij}'s. From Eq. (3), the synchronized chaotic state is stable if $0.35 < a_{ij} < 1.65$. We have numerically verified this result.

If we have nearest neighbor coupling we can obviously obtain better bounds than given in Eq. (3) since we know more information about the coupling coefficients. In this case, we obtain the following bounds (further details on this and other coupling schemes can be found in our forthcoming paper [32]):

$$\frac{1 - \exp(-h_{max})}{1 - \cos(2\pi/L)} < a_{ij} < \frac{1 + \exp(-h_{max})}{1 + \cos(2\pi/L)}, \quad \text{if } L \text{ is odd,}$$
$$\frac{1 - \exp(-h_{max})}{1 - \cos(2\pi/L)} < a_{ij} < \frac{1 + \exp(-h_{max})}{2}, \quad \text{if } L \text{ is even.}$$

Note that as L becomes larger, the above range shrinks to zero. We find [32] a conservative estimate for the critical value L_c which makes the range zero to be:

$$L_c = \text{int}\left(\frac{2\pi}{\cos^{-1}\left[\exp(-h_{max})/(1+\exp(-h_{max}))\right]}\right). \tag{24}$$

This generalizes the result found in Ref. [11]. The above result implies that for a sufficiently large L ($> L_c$) nearest neighbor coupled systems can not have a stable synchronized chaotic state.

We now prove Theorem 2. The initial treatment is similar to the one used by Pecora and Carroll in arriving at the master stability function [30]. The structure of coupling that we have assumed includes the commonly used diffusive coupling, nearest neighbor coupling, all-to-all coupling, star coupling etc. Our proof breaks down if only one or few of the components of $\mathbf{x}^i$ are coupled. In this case, Pecora and Carroll [30] have shown numerically that more complicated stability conditions arise. Note that, unlike the coupled map case, we only have a lower bound in the stability condition. This difference arises from the fact that it is only for maps the stability condition is in terms of the absolute value of the eigenvalue which leads to both lower and upper bounds. This is not so for coupled differential equations.

The proof of this theorem is along the same lines as our proof for coupled maps. Linearizing around the synchronized state we get

$$\dot{\mathbf{z}}^i = \mathbf{J}(\mathbf{x})\mathbf{z}^i + \sum_{j\neq i} b_{ij}\mathbf{z}^j, \tag{25}$$

where $\mathbf{z}^i$ denotes deviations from $\mathbf{x}$ and $\mathbf{J}$ is the $M \times M$ Jacobian matrix. We now introduce the $M \times L$ state matrix $\mathbf{S} = (\mathbf{z}^1\ \mathbf{z}^2\ \ldots\ \mathbf{z}^L)$. Then the linearized equation Eq. (25) can be written as the following matrix equation:

$$\dot{\mathbf{S}} = \mathbf{J}(\mathbf{x})\mathbf{S} + \mathbf{S}\mathbf{C}^T, \tag{26}$$

where $\mathbf{C}^T$ is the transpose of the $L\times L$ coupling matrix $\mathbf{C}$ containing the coupling coefficients:

$$\begin{aligned} [\mathbf{C}]_{ii} &= -\sum_{j\neq i} b_{ij}, \quad i = 1, 2, \ldots, L, \\ [\mathbf{C}]_{ij} &= b_{ij}, \quad i \neq j \end{aligned} \tag{27}$$

Since the coupling is symmetric, $\mathbf{C}$ is Hermitian. Following the same procedure as before, we obtain

$$\dot{\mathbf{u}} = \left[\mathbf{J}(\mathbf{x}) + \lambda\mathbf{I}\right]\mathbf{e}, \tag{28}$$

where $\mathbf{I}$ is the $M\times M$ identity matrix, $\mathbf{e}$ is an eigenvector of $\mathbf{C}$ and λ its associated eigenvalue. We now compute the Lyapunov exponents for the above reduced system. We note that $\lambda = 0$ is always an eigenvalue of $\mathbf{C}$ and its corresponding eigenvector is $(1\ 1\ \ldots 1)^T$. This corresponds to the synchronized chaotic state. The Lyapunov exponents in this case are nothing but the Lyapunov exponents for the individual system. Hence they are given by $h_1 = h_{max}$, h_2, ... , h_M.

Next we consider the remaining eigenvalues and eigenvectors. Since $\mathbf{C}$ is a symmetric matrix, the remaining eigenvectors span a $(L-1)$-dimensional subspace orthogonal to the synchronization manifold. For each $\lambda \neq 0$ we calculate the Lyapunov exponents for Eq. (28). Since λI commutes with $\mathbf{J}(\mathbf{x})$, the Lyapunov exponents are easily calculated. Denoting them by $\mu_1(\lambda)$, $\mu_2(\lambda)$, ..., $\mu_M(\lambda)$, we have

$$\mu_i(\lambda) = h_i + \lambda, \quad i = 1, 2, \dots, M. \tag{29}$$

For stability, we require these transversal Lyapunov exponents for each $\lambda \neq 0$ to be negative. This is equivalent to the statement

$$\mu_{max}(\lambda) = h_{max} + \lambda < 0. \tag{30}$$

In other words, we require $\lambda < -h_{max}$ for each $\lambda \neq 0$. Ordering the eigenvalues of $\mathbf{C}$ as $\lambda_1 = 0 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_L$, the stability conditions can be rewritten as

$$\lambda_2 < -h_{max}. \tag{31}$$

As before, we wish to obtain bounds on b_{ij} such that the above inequality is satisfied. Consider a Hermitian matrix $\mathbf{K}'$ defined as follows:

$$\begin{aligned} [\mathbf{K}']_{ii} &= -(L-1)R', \quad i = 1, 2, \dots, L, \\ [\mathbf{K}']_{ij} &= R', \quad i \neq j, \end{aligned} \tag{32}$$

where R' is a constant which will be characterized later. Consider the matrix $\mathbf{P}' = \mathbf{K}' - \mathbf{C}$. We see that the diagonal elements are given by

$$\mathbf{P}'_{ii} = \sum_{j \neq i} b_{ij} - (L-1)R', \quad i = 1, 2, \dots, L. \tag{33}$$

These are positive if $b_{ij} > R' \; \forall \, i, j, \; j \neq i$. Next consider the absolute value of the off-diagonal elements of $\mathbf{P}'$:

$$|\mathbf{P}'_{ij}| = |R' - b_{ij}|, \quad \forall \, i, j, \; j \neq i. \tag{34}$$

If $b_{ij} > R'$, it can be shown that

$$|\mathbf{P}'_{ii}| \geq \sum_{j \neq i} |\mathbf{P}'_{ij}|, \quad i = 1, 2, \dots, L \tag{35}$$

and $\mathbf{P}'_{ii} > 0$. This implies that $\mathbf{P}'$ is positive semidefinite [31].

We have shown that $\mathbf{P}' = \mathbf{K}' - \mathbf{C}$ is positive semidefinite if $b_{ij} > R' \; \forall \, i, j \; j \neq i$. Since $\mathbf{K}'$ and $\mathbf{C}$ are Hermitian, we have $\mathbf{K}' \succeq \mathbf{C}$. Now the largest eigenvalues of both $\mathbf{K}'$ and $\mathbf{C}$ are equal to 0. The second largest eigenvalue of $\mathbf{K}'$ can be easily calculated and is equal to $-LR'$. Therefore $\mathbf{K}' \succeq \mathbf{C}$ implies that

$$\lambda_2 \leq -LR'. \tag{36}$$

Comparing with the inequality given in Eq. (31), we see that this constraint is obeyed if $-LR' < -h_{max}$. That is,

$$R' > h_{max}/L. \tag{37}$$

But $\lambda_2 \leq -LR'$ only if $b_{ij} > R' \; \forall \, i, j \; j \neq i$. Putting the two inequalities together, we get the required stability condition given in Eq. (4).

III. CONCLUSIONS

To conclude, we have derived very simple bounds on the coupling coefficients which ensure the stability of the synchronized chaotic state of L symmetrically coupled systems. We also gave specific bounds for the nearest neighbor coupled map system. These results allow for non equal coupling coefficients and generalize earlier results found in the literature [11,13] which were obtained assuming that all coupling coefficients are constant. It is very easy to apply our criteria to the system being studied and they encompasses a wide class of coupling schemes including most of the popularly used ones in the literature. Further, we expect the introduction of non-equal coupling to lead to interesting new phenomena in coupled systems. Our stability results would enable a systematic exploration of such systems.

Our results were made possible by a sequence of operations. We summarize them below since we feel that this approach is applicable to the stability analysis of a wide variety of coupled systems and not merely the specific problem considered in this letter. First, we converted the linearized system to a matrix equation. This was further simplified by looking at the evolution of eigenmodes of the coupling matrix $\mathbf{C}$. By realizing that only the largest Lyapunov exponent matters for our analysis, the stability conditions for the synchronized chaotic state were recast as bounds on certain eigenvalues of the coupling matrix. Then we bound the Hermitian matrix $\mathbf{C}$ in the coupled map system and the coupled oscillator system by carefully constructed constant Hermitian matrices $\mathbf{K}$ and $\mathbf{K}'$ respectively. This was done in such a manner that that $\mathbf{K} - \mathbf{C}$ and $\mathbf{K}' - \mathbf{C}$ are positive semidefinite when the coupling coefficients satisfy certain inequalities. Using a powerful result from matrix analysis, this automatically implies that the eigenvalues of $\mathbf{C}$ are bounded above by the eigenvalues of $\mathbf{K}$ (or $\mathbf{K}'$). By comparing the different bounds that we had derived, we then finally arrived at Eqs. (3) and (4) which ensure the stability of the synchronized chaotic state in the coupled map and coupled oscillator systems respectively.

ACKNOWLEDGEMENTS

The work was supported by US ONR Grant N00014-99-1-0062. GR thanks Center for Complex Systems and Brain Sciences, Florida Atlantic University, where this work was performed, for hospitality. He is also associated with the Jawaharlal Nehru Center for Advanced Scientific Research as a honorary faculty member and supported by the Homi Bhabha Fellowship and DRDO and ISRO, India as part of the Nonlinear Studies Group. We thank the referees for their useful comments.

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