Chapter 25

Applications of Chaos

In this chapter applications of the theoretical understanding of chaos are considered. Many of the ideas are quite straightforward applications of the theory developed in previous chapters, and the reader is referred to the original papers for further information.

25.1 Targeting

The targeting of chaotic trajectories uses the sensitive dependence on initial conditions to produce large (desired) changes to the trajectory using small perturbations. One important application is to directing spacecraft using very small amounts of fuel, relying on the (Hamiltonian) chaos of the three body problem consisting of the spacecraft, earth, and moon or other planet. Another application is to improving the control of chaos, and we will establish the concept in this context.

One problem with the methods of controlling chaos described in the previous chapter is that they require the trajectory to approach the control point to within some small tolerance ε before the control method can be applied. The number of iterations for this approach can be estimated as $\tau_0 \sim 1/\mu(\varepsilon)$ with $\mu(\varepsilon)$ the natural measure of the attractor associated with the ε-ball. The measure of a small ball at a point x scales with the ball size according to the pointwise dimension $D_P(x)$ which can be estimated as the information dimension D_1, so that $\tau_0 \sim \varepsilon^{-D_1}$. This power law dependence on ε can lead to large capture times, especially for high dimensional chaos. The method of targeting can be used to convert this to a much weaker logarithmic dependence.

Suppose we are starting from a point x_0 and wish to control the system at a point x_f. The idea of the scheme is to iterate forwards from a small ball of initial conditions of size δ surrounding x_0. By the sensitivity to initial conditions the dimensions of this ball will rapidly grow in the expanding dimensions to become comparable with the extent of the attractor in a few iterations. Similarly iterate the ε-ball around x_f *backwards* until its extent becomes comparable to the size of the attractor. Look for an intersection between the forward and backward iterated balls, and use this to calculate the perturbation at x_0 that will lead to the trajectory passing through this point and therefore arriving within the ε-ball at x_f. Consider the case of a two dimensional map, with a chaotic attractor with positive Lyapunov exponent λ_u and negative exponent λ_s. Under the forward iteration the ball grows at rate λ_u along the unstable direction; under backward iteration the ball grows at rate $|\lambda_s|$ along the stable direction of the attractor. The number of iterations required, estimated as the sum of the times for these dimensions to get to $O(1)$ is

$$\tau \sim \lambda_u^{-1} \ln(1/\delta) + |\lambda_s|^{-1} \ln(1/\varepsilon), \tag{25.1}$$

i.e. with a logarithmic dependence on the control range ε. For the extension to higher dimensions and the presence of noise or modeling imperfections, and the implementation for the Hénon map, see Shinbrot et al. [1].

It is now well known (e.g. freshman physics) that spacecraft can be accelerated using close swing-bys of the moon or planets. Combining this with the sensitive dependence of the trajectories to small perturbations due to chaos in the three body problem allows, in favorable situations, the use of small thrusts to direct spacecraft to the vicinity of specific objects in the solar system. Aligood et al ("Lab visit 2") describe this method that was used by NASA in 1982 to redirect a spacecraft to visit the Giacobini-Zinner comet using 37 small thrust burns and 5 lunar swing-bys. Schroer and Ott [2] describe the application to Earth-Moon trajectories. They also describe enhancements to the "forward-backward" approach that recognizes the existence of bottlenecks for trajectories passing between different regions of the chaotic orbits. (These bottlenecks can be identified using the idea of resonances between periodic orbits.) The forward-backward method is enhanced by segmenting the trajectory between these known bottlenecks.

25.2 Synchronization

Suppose we have two *identical* chaotic systems, for example each described by three ordinary differential equations, for variables $x^{(1)}, y^{(1)}, z^{(1)}$ and $x^{(2)}, y^{(2)}, z^{(2)}$

$$\begin{aligned} \dot{x}^{(i)} &= X(x^{(i)}, y^{(i)}, z^{(i)}) \\ \dot{y}^{(i)} &= Y(x^{(i)}, y^{(i)}, z^{(i)}) \\ \dot{z}^{(i)} &= Z(x^{(i)}, y^{(i)}, z^{(i)}) \end{aligned} \tag{25.2}$$

with the functions X, Y, Z the same for both systems. Now suppose for system 2 we replace $x^{(2)}$ in Y, Z by $x^{(1)}$ (and then the equation for $\dot{x}^{(2)}$ is not needed). Starting from arbitrary initial conditions it will often be found that the dynamics of 2 becomes *synchronized* to 1 i.e. the second system tracks the first one, $|y^{(2)} - y^{(1)}| \to 0$ and $|z^{(2)} - z^{(1)}| \to 0$ as $t \to \infty$ [3],[4]. The possibility of synchronization can be analyzed by considering the growth of perturbations about the synchronized orbit in the restricted tangent space of perturbations $\delta y^{(2)}, \delta z^{(2)}$ about the orbit $x^{(1)}(t), y^{(2)}(t), z^{(2)}(t)$. This can be used to define two "conditional Lyapunov exponents": these must both be negative for the synchronization to be stable. (This analysis does not of course address the question of attraction to the synchronized orbit from arbitrary initial conditions.) For example Pecora and Carroll [3] considered the Lorenz model (with $r = 60$, $b = 8/3$ and $\sigma = 10$) and the Rossler model (with $a = 9.0$, $b = 0.2$, and $c = 0.2$, in my notation), and found the following results:

System	Drive	Response	Lyapunov Exponents	Synchronization?
Rossler	x	y,z	0.2 −8.89	No
	y	x,z	−0.056 −8.81	Yes
	z	x,y	0.1 0.1	No
Lorenz	x	y,z	−1.81 −1.86	Yes
	y	x,z	−2.67 −9.99	Yes
	z	x,y	0.011 −11.0	No

The synchronization seems to be robust, in the sense that if the parameters of the two systems differ slightly, the orbits remain close to one another.

The reason synchronization does not occur in the last entry in the table can be readily understood. The equations for the driven system are

$$\begin{aligned} \dot{x}^{(2)} &= -\sigma(x^{(2)} - y^{(2)}) \\ \dot{y}^{(2)} &= rx^{(2)} - y^{(2)} - x^{(2)}z^{(1)} \end{aligned} . \tag{25.3}$$

with $z^{(1)}$ the drive from system 1. These are homogeneous in $x^{(2)}$, $y^{(2)}$ so for any solution $x^{(2)}(t)$, $y^{(2)}(t)$ then $kx^{(2)}(t)$, $ky^{(2)}(t)$ is also a solution. This suggests the maximum exponent should be 0, and not .01 as listed in the table [5].

Many other synchronization schemes are possible. For example one natural scheme is a proportional coupling

$$\frac{d\vec{x}^{(1)}}{dt} = \vec{F}(x^{(1)}) \quad \frac{d\vec{x}^{(2)}}{dt} = \vec{F}(x^{(2)}) + \vec{B}(v-u) \tag{25.4}$$

where

$$u = \vec{K}^T \cdot \vec{x}^{(1)} \quad v = \vec{K}^T \cdot \vec{x}^{(2)} \tag{25.5}$$

where the synchronization "signal" is the single variable u given by a linear combination of the components of $\vec{x}^{(1)}$ determined by $\vec{K}$, and $\vec{B}$ gives the influence of the signal in each dynamical equation of system 2. The possibility of synchronizing $\vec{x}^{(2)} = \vec{x}^{(1)}$ is then given by the properties of the tangent space of $\vec{x}^{(2)}$ defined by the linear operator $A\left(\vec{x}^{(1)}(t)\right) - \vec{B}\vec{K}^T$ with A the Jacobean $\partial\vec{F}/\partial\vec{x}^{(2)}$ evaluated at $\vec{x}^{(2)} = \vec{x}^{(1)}(t)$. The analogies with control theory are now apparent, except we are "controlling" to the chaotic state $\vec{x}^{(1)}(t)$ rather than to a fixed point, and so A is not a constant matrix. One result to be expected then is that a single control signal u may be sufficient to synchronize chaotic attractors with more than one unstable direction. The generalization to an M dimensional vector of synchronization signals $\vec{u}$ (when B and K become $N \times M$ matrices as in the general control theory) is also straightforward.

The synchronization of chaos has been proposed as an analog encryption scheme for communication channels (for a version using a circuit implementing the Lorenz equations see [6]). The idea is that the data is hidden under a noisy carrier generated by a chaotic system, such as a chaotic electronic circuit. At the receiving end the data is regenerated by synchronizing a similar chaotic system to the signal, and looking at the error. However, since the properties of low dimensional chaotic systems can be reconstructed by analysis of the signal, such as through the delay time reconstruction method (see chapter 11), it seems unlikely that these systems might provide a secure encryption method.

25.3 Generalized Synchronization

The idea of generalized synchronization is that two *different* chaotic systems may become synchronized in the sense that the variables of the driven system are some unique function of the variables of the other system $\vec{x}^{(2)} = \vec{\phi}(\vec{x}^{(1)})$ at long times [7]. The synchronization of two slightly different systems above is an example of generalized synchronization.

The condition for generalized synchronization for the "one-way coupled" system of drive ($\vec{x}^{(1)}$) and response ($\vec{x}^{(2)}$) variables

$$\frac{d\vec{x}^{(1)}}{dt} = \vec{F}(\vec{x}^{(1)}) \tag{25.6}$$

$$\frac{d\vec{x}^{(2)}}{dt} = \vec{G}(\vec{x}^{(2)}, \vec{h}(\vec{x}^{(1)})) \tag{25.7}$$

is the following [5],[8]:

> Generalized synchronization occurs if and only if for all initial $\vec{x}^{(1)}(0)$ in a neighborhood of the chaotic attractor of the drive system, the response system is asymptotically stable. Asymptotic stability is the notion that all trajectories $\vec{x}^{(2)}(t)$ from a range of initial conditions of $\vec{x}^{(2)}$ converge to a unique trajectory at long times, i.e.
>
> $$\lim_{t\to\infty} \left| \vec{x}^{(2)}(t; \vec{x}^{(1)}(0), \vec{x}^{(2)}(0)) - \vec{x}^{(2)}(t; \vec{x}^{(1)}(0), \vec{x}^{(2)\prime}(0)) \right| = 0 \tag{25.8}$$
>
> where $\vec{x}^{(2)}(0)$ and $\vec{x}^{(2)\prime}(0)$ are two different initial conditions for the driven system.

Since this condition is quite restrictive, the examples of generalized synchronization that have been demonstrated are either the synchronization to the same chaotic system with slightly different parameters, or the synchronization of systems where the response system is linear (or is a "hidden" linear system given by a trivial nonlinear transformation of a linear system). An example [5] is the synchronization of a "Lorenz" system

$$\begin{aligned}
\dot{x}^{(2)} &= -\sigma \left(x^{(2)} - y^{(2)} \right) \\
\dot{y}^{(2)} &= r u^{(1)}(t) - y^{(2)} - u^{(1)}(t) z^{(2)} \\
\dot{z}^{(2)} &= u^{(1)}(t) y^{(2)} - b z^{(2)}
\end{aligned} \tag{25.9}$$

with a chaotic Rossler system

$$\begin{aligned} \dot{x}^{(1)} &= 2 + x^{(1)}(y^{(1)} - 4) \\ \dot{y}^{(1)} &= -x^{(1)} - z^{(1)} \\ \dot{z}^{(1)} &= y^{(1)} + 0.45z^{(1)} \end{aligned} \tag{25.10}$$

where $u^{(1)}$ is any function of $x^{(1)}, y^{(1)}, z^{(1)}$. Note that in all the nonlinear (quadratic) terms of the Lorenz system, one of the $\vec{x}^{(2)}$ variables is replaced by the system 1 variable $u^{(1)}$ so that these equations become *linear* in $\vec{x}^{(2)}$. If we define a difference variable between solutions from two different initial conditions for the driven system $e_x = x_1^{(2)} - x_2^{(2)}$, etc. it is easily checked that the function

$$L = (e_x^2/\sigma + e_y^2 + e_z^2) \tag{25.11}$$

satisfies $\dot{L} < 0$ for any non-zero e_x, e_y, e_z, so that e_x, e_y, e_z necessarily approach zero at long times. Identifying such a function (called a Lyapunov function) is one of the few ways of proving asymptotic stability, and this is certainly easiest to do for a linear system of equations.

For a linear response system it is perhaps no surprise that there is a unique relationship between $\vec{x}^{(2)}$ and $\vec{x}^{(1)}$ (since $\vec{x}^{(1)}(t)$ implicitly defines all $\vec{x}^{(1)}(t')$ for $t' < t$ that are needed for the explicit formal integration of the linear system 25.9). This naive argument suggests that the response system must be sufficiently contracting so that only a reasonably short prehistory $t' < t$ of $\vec{x}^{(1)}(t)$ is needed, for otherwise the uncertainty in constructing the prehistory due to the divergence of trajectories integrating *backwards* will be important. If this is not the case the function $\vec{\phi}$ will not be smooth and in fact develops fractal structure, so that the construction is presumably less useful—for example the dimension of the combined system, or the dimension given by the delay-coordinate reconstruction from measurements of $\vec{x}^{(2)}(t)$, will no longer match that of the drive system.

Hunt et al. [8] show that the (sufficient) criterion for the transformation $\vec{\phi}(\vec{x}^{(1)})$ to be smooth (differentiable) is $h_r(\vec{x}^{(1)}) > h_d(\vec{x}^{(1)})$, where $-h_r(\vec{x}^{(1)})$ is the least-negative response system "past-history" Lyapunov exponent and $-h_d(\vec{x}^{(1)})$ is the most-negative drive system "past-history" Lyapunov exponent. (The past-history Lyapunov exponent is the Lyapunov exponent constructed from infinitely long trajectories that *end* at $\vec{x}^{(1)}$. Note that here we may be interested in the rare set of $\vec{x}^{(1)}$ that have anomalous values of the Lyapunov exponents since these may lead to singularities of the map function $\vec{\phi}$ at particular $\vec{x}^{(1)}$, rather than to the typical $\vec{x}^{(1)}$ that lead to the "average" Lyapunov exponents.) This paper presents results for ϕ when the drive system $(x^{(1)}, y^{(1)})$ is the bakers map (see chapter 5), and the response system is the simple one dimensional linear system

$$x_{n+1}^{(2)} = \lambda x_n^{(2)} + x_n^{(1)}. \tag{25.12}$$

25.4 Predicting A from B

Often it might be useful to be able to evaluate a variable A from measurements of a different variable B, without knowing any theoretical relationship between the two variables. For example the variable B may be easily accessible, but the variable A much less so. On a chaotic attractor the idea of embedding can be used to do this, and this has been called "predicting A from B". Again the basic idea is straightforward, although various ingenious tricks might help in the implementation [9]. If the dimension of the attractor is D then the dynamics can be reconstructed using measurements of the single variable $\{B(t - j\tau)\}$, $j = 0 \ldots n-1$, with some suitable choice of the delay time τ and embedding dimension n related to D. The set $A(t), \{B(t - j\tau)\}$ must then be redundant, so that $A(t)$ can be predicted $A(t) = \phi(\{B(t - j\tau)\})$. The map function ϕ is not known *a priori*, but is determined numerically in a learning phase (e.g. via a local polynomial fit). Various wrinkles in the implementation are to use a time symmetric embedding i.e. predict $A(t)$ from $B(t + j\tau)$, $j = -n/2 \ldots n/2$ since the correlations will be better over a shorter time and to use the full embedding dimension $2D + 1$ to avoid spurious near neighbors.

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